Dynamical Behavior Analysis of Fixed Points of Investment Competition Model

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Abstract. First, based on Logistic map, a duopoly investing model with discrete formula is constructed for the homogeneous firms. Then, the long-time evolution processes of the model are analyzed including the stability of fixed points, bifurcations and chaotic behaviors. Moreover, the economic implication characterized the dynamics behaviors are indicated. The results show that in some extent of parameter, the final results of the model are stable points, and in other parameter extent the final results are quasi-periodic motion or chaotic attractor. It's interesting that with the increase of parameter chaos induced by double periodic bifurcation replaced by stable equilibrium. Finally, numerical simulations are presented to verify the results.

Keywords: dynamics, duopoly investing model, bifurcation, chaos.

1 Introduction

The literatures concerning the economics and finance via dynamical models are voluminous in recent years. A lot of economic models, such as fighting for few game in finance market[1, 2], Krishnamurthy's model in advertising competition[3, 4], Cournot duopoly model in investment competition[5–10], and a lot of cooperation and competition model of duopoly[11] have been studied by economists, managers and applied mathematicians. In those models, certain evolution equations are used to describe various long-time states characterized by equilibrium points, bifurcations, limit cycles and chaotic attractors. The competitive strength and economic power of two firms, which are well-matched in economic scale, productivity, and the level of technology, will greatly change in a short period. Sequentially, one may be the leader in a certain industry, and the other may be atrophied and withdraw from the market. To make themselves fully developed, each firm suppresses the rival, captures market share and increases investment.

Xiaoshu Luo in [12] analyzes an investment competition model, which presents complex dynamical behaviors. Here our concern is to give a modified model of

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that in [12], which is based on Logistic map, describing investment competition of two firms, and discussing it's own fixed points and their complex dynamical behaviors via the parameters.

This paper is organized as follows. Section 2 constructs the model of nonlinear difference equation. Section 3 solves the fixed points of the system and analyzes their dynamical behaviors. Section 4 gives some examples to illustrate the theoretical results. Finally, Section 5 concludes the paper.

2 Construction of the Model

We consider two firms, $i = 1, 2$, producing the same good for sale in the market. Investment decisions of both firms occur in discrete time periods, $n = 1, 2, \cdots$. x_n, y_n are the amounts of investment of the two firms respectively at time period n, $0 \le x_n, y_n \le 1$. According to their economical meanings[13], we can suppose that the amount of investment of firm 1, x_{n+1}, at next time period is directly influenced by the amount of investment of it's rival at this stage y_n. Meanwhile, $1-y_n$ is also a considerable factor for the total demand of the market is relatively unchanged. Besides, x_{n+1} is affected by x_n, which is called inertial effect, and on the other side, x_{n+1} is constrained by the own's amount of investment at upper stage, as resource constraints, internal trade union opposition. Let $1-\gamma$ is the inertial factor, we can construct the model of two enterprises competition investment, which can be described by

$$\begin{cases} x_{n+1} = (1-\gamma)\mu x_n(1-x_n) + \gamma\mu y_n(1-y_n) \\ y_{n+1} = (1-\gamma)\mu y_n(1-y_n) + \gamma\mu x_n(1-x_n) \end{cases} \tag{1}$$

here $0 < \gamma < 1$, $0 < \mu \le 4$. It can be known that this model is based on Logistic map which is widely applied in ecological system.

3 Analyzing Dynamics of Fixed Points

In order to study the qualitative behavior of the solutions of the duopoly investing model (1), solving the nonnegative fixed point of equation (1) we get its equilibrium points $(s_i, t_i), i = 1, 2, 3, 4$, here $s_1 = t_1 = 0$; $s_2 = t_2 = 1 - \frac{1}{\mu}$ if $\mu > 1$; and

$$s_{3,4} = \frac{1}{2} - \frac{1}{2\mu} + \frac{1}{\mu}\frac{\gamma}{2\gamma-1} \mp \frac{1}{2\mu}\sqrt{(\mu-1)^2 - \frac{4\gamma^2}{(2\gamma-1)^2}},$$

$t_3 = s_4, t_4 = s_3$, if $\frac{2}{3} < \gamma < 1$, and $\mu_1^* = 1 + \frac{2\gamma}{2\gamma-1} < \mu \le 4$.

3.1 Stability Analysis of Fixed Points

Let $f(x) = \mu x(1-x)$, then $f'(x) = \mu(1-2x)$. Jacobin matrix of system (1) at fixed point $P(s, t)$ is

$$J = \begin{bmatrix} (1-\gamma)f'(s) & \gamma f'(t) \\ \gamma f'(s) & (1-\gamma)f'(t) \end{bmatrix}$$

The eigenpolynomial of J is

$$|\lambda I - J| = [\lambda - (1-\gamma)f'(s)][\lambda - (1-\gamma)f'(t)] - \gamma^2 f'(s)f'(t). \tag{2}$$

Via nonlinear dynamics[14], we know that when the eigenvalues $\lambda_i, (i = 1, 2)$ satisfied $|\lambda_{1,2}| \leq 1$ and when the spectral radius of J is $\rho(J) = 1$, $\lambda_1 = 1$ is a simple elementary divisor, and $|\lambda_2| < 1$, the fixed point (s, t) is stable; and when $\lambda_1 = -1$, or $\lambda_2 = -1$, at point (s, t), double period bifurcation occurs.

1) Stability analysis of fixed point P_1

Substitute s, t in eigenpolynomial (2) with s_1 and t_1 respectively, we can get the eigenvalues $\lambda_1 = \mu$, $\lambda_2 = \mu(1-2\gamma)$. Considering $0 < \gamma < 1$, so $-1 < 1-2\gamma < 1$, we can make $|\lambda_{1,2}| < 1$ by choosing $0 < \mu < 1$. When $\mu = 1$, we have $\lambda_1 = 1$ and $-1 < \lambda_2 < 1$. Therefor when $0 < \mu \leq 1$, the fixed point P_1 is stable.

2) Stability analysis of fixed point P_2

Substitute s, t in eigenpolynomial (3) with s_2 and t_2 respectively, we can get the eigenvalues $\lambda_1 = 2 - \mu$, $\lambda_2 = (2 - \mu)(1 - 2\gamma)$. We can make $|\lambda_{1,2}| < 1$ by choosing $-1 < 2 - \mu < 1$, i.e. $1 < \mu < 3$. Therefor when $1 < \mu < 3$, the fixed point P_2 is stable; When $\mu = 3$, $\lambda_1 = -1$, double period bifurcation will occur.

3) Stability analysis of fixed point P_3, and P_4

Because P_3, and P_4 correspond to the same eigenpolynomial, we only discuss the stability of P_3. The situation of P_4 is the same. Whereas it must be noticed that P_3, and P_4 have different domain of attraction.

Substituting s, t in eigenpolynomial (3) with s_3 and t_3 respectively, we can get the eigenvalues

$$\lambda_{1,2} = -\frac{1-\gamma}{2\gamma-1} \pm \sqrt{\frac{(1-\gamma)^2}{(2\gamma-1)^2} + (2\gamma-1)[\frac{1+4\gamma^2}{(2\gamma-1)^2} - (\mu-1)^2]},$$

To make $|\lambda_{1,2}| < 1$, we divide it into two conditions,

$$\begin{cases} \frac{(1-\gamma)^2}{(2\gamma-1)^2} + (2\gamma-1)[\frac{1+4\gamma^2}{(2\gamma-1)^2} - (\mu-1)^2] \geq 0 \\ \frac{1-\gamma}{2\gamma-1} + \sqrt{\frac{(1-\gamma)^2}{(2\gamma-1)^2} + (2\gamma-1)[\frac{1+4\gamma^2}{(2\gamma-1)^2} - (\mu-1)^2]} < 1 \end{cases} \tag{3}$$

and

$$\begin{cases} \frac{(1-\gamma)^2}{(2\gamma-1)^2} + (2\gamma-1)[\frac{1+4\gamma^2}{(2\gamma-1)^2} - (\mu-1)^2] < 0 \\ |\lambda_{1,2}|^2 = -(2\gamma-1)[\frac{1+4\gamma^2}{(2\gamma-1)^2} - (\mu-1)^2] < 1 \end{cases} \tag{4}$$

From (3), we have

$$\begin{cases} \mu_2^* < \mu \leq \mu_3^* \\ \frac{2}{3} < \gamma < 1, \end{cases} \tag{5}$$

here $\mu_2^* = 1 + \sqrt{1 + \frac{3}{(2\gamma-1)^2}}$, $\mu_3^* =\leq 1 + \sqrt{\frac{4\gamma^2}{(2\gamma-1)^2} + \frac{\gamma^2}{(2\gamma-1)^3}}$. We can verify $\mu_1^* < \mu_2^*$. And it can be made $\mu_2^* < 4$ by choosing $g(\gamma) = 32\gamma^2 - 32\gamma + 5 > 0$, which equivalents to $\gamma_1^* < \gamma < 1$, here $\gamma_1^* = \frac{\sqrt{6}+4}{8} \approx 0.8062$. And it can also be made $\mu_3^* \leq 4$ by choosing $h(\gamma) = 64\gamma^3 - 105\gamma^2 + 54\gamma - 9 \geq 0$, which equivalents to $\gamma_2^* \leq \gamma < 1$, here γ_2^* is the root of $h(\gamma) = 0$ and $\frac{2}{3} < \gamma_2^* < 1, \gamma_2^* \approx 0.8231$.

From (4), we have

$$\begin{cases} \mu_3^* < \mu < \mu_4^* \\ \gamma_2^* < \gamma < 1, \end{cases} \tag{6}$$

here $\mu_4^* = 1 + \frac{\sqrt{2\gamma(2\gamma+1)}}{2\gamma-1}$. It can be made $\mu_4^* \leq 4$ by choosing $k(\gamma) = 32\gamma^2 - 38\gamma + 9 \geq 0$, which equivalents to $\gamma_3^* \leq \gamma < 1$, here $\gamma_3^* = \frac{\sqrt{292}+19}{32} \approx 0.8608$.

So when

$$\begin{cases} \mu_2^* < \mu < \mu_4^* \\ \gamma_3^* \leq \gamma < 1 \end{cases} \tag{7}$$

the fixed point P_3 is stable; when $\mu = \mu_2^*$, $\lambda_2 = -1$, double period bifurcation will occur.

3.2 Bifurcation Analysis of Fixed Points P_3, P_4

To analyze the bifurcation of P_3, P_4, after they are unstable, we discuss some properties of the fixed points P_3, P_4 furthermore. When the parameters γ, and μ are satisfied with the formula (5), the eigenvalues $\lambda_{1,2}$ are two real roots, and the fixed points P_3, P_4 are stable nodes; when the parameters γ, and μ are satisfied with the formula (6), the eigenvalues $\lambda_{1,2}$ are conjugate complex roots and the fixed points P_3, P_4 are stable focuses.

For 2-dim discrete system, when the stable focus is unstable, if the eigenvalues $\lambda_{1,2}$ of the Jacobin matrix at the fixed point satisfy one of the following conditions

$$|\lambda_1(\mu^*)| = 1, \frac{\partial|\lambda_1(\mu)|}{\partial\mu} |_{\mu=\mu^*} > 0, \tag{8}$$

super-critical Hopf bifurcation will occur at this fixed point[14]. Here $\mu^* = \mu_4^*$, $\lambda_{1,2}$ are conjugate complex roots. So we get $|\lambda_1(\mu)|^2 = -\frac{1+4\gamma^2}{2\gamma-1} + (2\gamma-1)(\mu-1)^2$ and $\frac{\partial|\lambda_1(\mu)|^2}{\partial\mu} = 2(2\gamma-1)(\mu-1)$ so $\frac{\partial|\lambda_1(\mu)|}{\partial\mu} |_{\mu=\mu_4^*} > 0$ In a short, the fixed points P_3, P_4 satisfy the condition of super-critical Hopf bifurcation, after they leave stable focuses, and they will behavior in quasi-period.

4 Numerical Simulation

In this section, we will perform some numerical simulations to verify the theoretical results obtained in previous sections. Fig.1-Fig.5 are bifurcation diagrams, phase diagrams and wave diagrams of system (1) at various initial conditions. And we also calculate the Lypunov exponents of system (1) which are represented in Fig.6.

When $\gamma = 0.85$, for $\gamma_2^* < \gamma < \gamma_3^*$ we can calculate $\mu_1^* = 3.4286$, $\mu_2^* = 3.6688$, $\mu_3^* = 3.8292$, $\mu_4^* = 4.0606 > 4$. From Fig.1 we know that when $\gamma = 0.85$, if $0 < \mu \leq 1$, P_1 is stable fixed point; if $1 < \mu < 3$ the fixed point P_1 is out-of-stable, the moving points converge on the fixed point P_2; if $3 \leq \mu \leq \mu_2^*$, the fixed

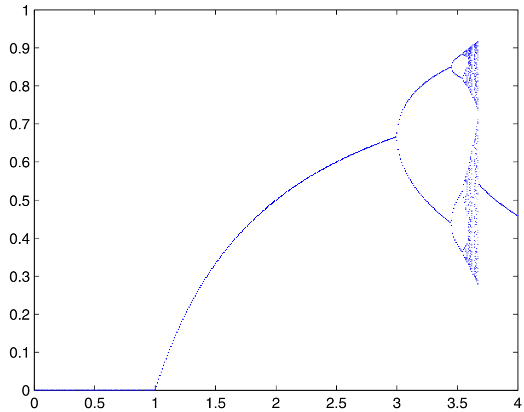

Fig. 1. Bifurcation diagram of variable x_n vs μ in system (1) with $\gamma = 0.85$, $x_0 = 0.46, y_0 = 0.87$

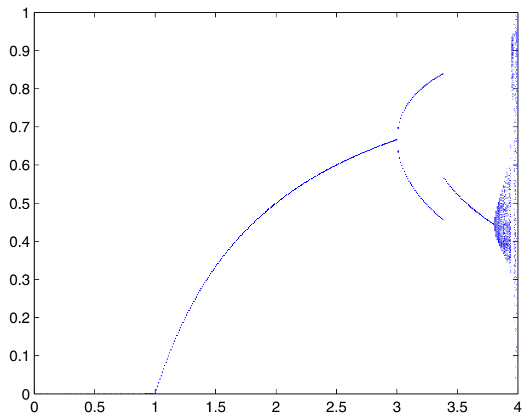

Fig. 2. Bifurcation diagram of variable x_n vs μ in system (1) with $\gamma = 0.90$, $x_0 = 0.46, y_0 = 0.87$

point P_2 is out-of-stable, double period bifurcation occurs; and if $\mu_2^* < \mu \leq 4$, the moving points converge on the fixed pointP_3 (or P_4). These diagrammatic results correspond to the theoretical ones. Fig.4 presents a chaotic attractor of system (1) for $\gamma = 0.85$, $\mu = 3.5706$ with two Lyapunov exponents as 0.0257, and -0.0038, initial condition $x_0 = 0.46, y_0 = 0.87$.

When $\gamma = 0.90$, for $\gamma_3^* < \gamma < 1$ we can calculate $\mu_1^* = 3.25$, $\mu_2^* = 3.3848$, $\mu_3^* = 3.5777$, $\mu_4^* = 3.8062 < 4$. From Fig.2 we know that when $\gamma = 0.90$, if $0 < \mu \leq 1$, P_1 is stable fixed point; if $1 < \mu < 3$ the fixed point P_1 is out-of-stable, the moving points converge on the fixed point P_2; if $3 \leq \mu \leq \mu_2^*$, the

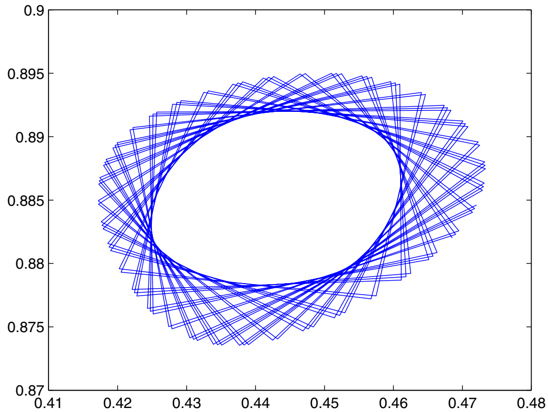

Fig. 3. Phase diagram x_n-y_n of quasi-periodic motion with $\gamma = 0.90$, $\mu = 3.85$, $x_0 = 0.46, y_0 = 0.87$

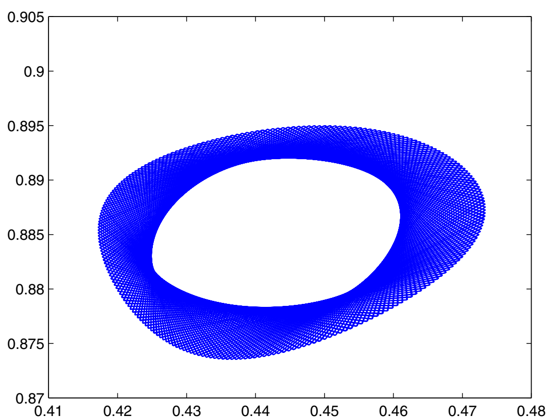

Fig. 4. Chaotic attractor x_n-y_n with one positive Lyapunov exponent for $\gamma = 0.85$, $\mu = 3.5706$, $x_0 = 0.46, y_0 = 0.87$

fixed point P_2 is out-of-stable, double period bifurcation occurs; if $\mu_2^* < \mu \leq \mu_4^*$, the moving points converge on the fixed point P_3 (or P_4), and if $\mu = \mu_4^*$, supercritical Hopf bifurcation occurs, and if $\mu_4^* \leq \mu \leq 4$, the moving points behavior in quasi-period. These diagrammatic results correspond to the theoretical ones. Fig.3 presents phase diagram of quasi-periodic motion with $\gamma = 0.90$, $\mu = 3.85$, initial condition $x_0 = 0.46, y_0 = 0.87$. Fig.5 presents a chaotic attractor of system (1) for $\gamma = 0.90$, $\mu = 3.960$ with two Lyapunov exponents as 0.3068, and 0.2878, initial condition $x_0 = 0.46, y_0 = 0.87$.

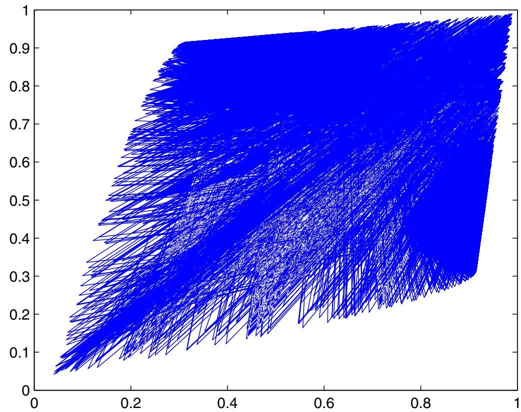

Fig. 5. Chaotic attractor x_n-y_n with two positive Lyapunov exponents for $\gamma = 0.90$, $\mu = 3.96$, $x_0 = 0.46, y_0 = 0.87$

5 Conclusions and Discussions

In this paper, based on Logistic map, we have constructed a discrete dynamical model of investment competition and got four fixed points. When $0 < \mu \leq 1$, the fixed point P_1 is stable, which means the two firms will disappear gradually in the market. When $1 < \mu < 3$, the fixed point P_2 is stable, and when $\mu = 3$, double period bifurcation occurs, which means the two firms will be synchronous gradually, and the game comes to a Nash equilibrium. When $\mu_2^* < \mu < \mu_4^*$, and $\gamma_3^* \leq \gamma < 1$, the fixed point P_3 is stable. When $\mu = \mu_2^*$, double period bifurcation occurs. When $\mu = \mu_4^*$, super-critical Hopf bifurcation occurs, and after leaving this focus point, the system behaves in quasi-period, and with the increase of parameter μ, the system comes into chaos. This result is unlike that in [12]. For parameter $\mu_2^* < \mu \leq \mu_4^*$, the evolution result of the system is a stable equilibrium again, P_3 or P_4. And for parameter $\mu_4^* < \mu \leq 4$, the super-critical Hopf bifurcation occurs, the investment amounts of two firms change through quasi-period trajectories into chaos.

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